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Positive solution for higher-order singular infinite-point fractional differential equation with p-Laplacian

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Abstract

By means of the method of upper and lower solutions together with the Schauder fixed point theorem, the conditions for the existence of at least one positive solution are established for some higher-order singular infinite-point fractional differential equation with p-Laplacian. The nonlinear term may be singular with respect to both the time and the space variables.

MSC: 26A33; 34B15; 34B16

Keywords: fractional differential equations; p-Laplacian; singularity; upper and lower solutions; positive solution

1 Introduction

We investigate the existence of positive solutions for the following fractional differential equations containing a p-Laplacian operator (PFDE, for short) and infinite-point boundary value conditions:

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ D_{0+}^{\alpha}u(0) = 0, \ u^{(i)}(1) = \sum_{j=1}^{\infty}\alpha_j u(\xi_j), \end{cases} \tag{1}$$

where D_{0+}^{α}, D_{0+}^{β} is the standard Riemann-Liouville derivative, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $f \in C((0,1) \times J, J)$, $J = (0,+\infty)$, $R^+ = [0,+\infty)$. $f(t,u)$ may be singular at $t = 0,1$ and $u = 0$, $i \in [1, n-2]$ is a fixed integer, $n-1 < \alpha \leq n$, $n \geq 3$, $0 < \beta \leq 1$, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ $(j = 1,2,\ldots)$, $\Delta - \sum_{j=1}^{\infty}\alpha_j \xi_j^{\alpha-1} > 0$, $\Delta = (\alpha-1)(\alpha-2)\cdots(\alpha-i)$.

In recent years, many excellent results of fractional differential equations have been widely reported for their numerous applications such as in electrodynamics of a complex medium, control, electromagnetic, polymer rheology, and so on; see [1–21] for an extensive collection of such results. In [4–6], by means of a fixed point theorem and the theory of the fixed point index together with the eigenvalue with respect to the relevant linear operator, the existence and multiplicity of positive solutions, pseudo-solutions are obtained for the m-point boundary value problem of the fractional differential equations

$$(A) \quad D_{0+}^{\alpha}u(t) + q(t)f\bigl(t,u(t)\bigr) = 0, \quad 0 < t < 1,$$

subject to the following boundary conditions:

$$(B_1) \quad u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u(1) = \sum_{i=1}^{m} \alpha_i u(\xi_i).$$

Similar results are extended to more general boundary value problems in [7]. Motivated by [8], by introducing height functions of the nonlinear term on some bounded sets, we considered the local existence and multiplicity of positive solutions for BVP (A) with infinite-point boundary value conditions in [9]. On the other hand, there have been some papers dealing with the fractional differential equations involving p-Laplacian operator [10–16]. The purpose of this paper is to study the existence of at least one positive solution for PFDE (1) by means of the upper and lower solutions and the Schauder fixed point theorem. A function $u \in C[0,1]$ is said to be a positive of problem (1) if $u(t) > 0$ on $t \in (0,1)$ and u satisfies (1) on $[0,1]$.

Compared to [5–7], this paper admits the following three new features. First of all, the fact that the p-Laplacian operator, involved in differential operator and infinite points, is contained in boundary value problems makes the problem considered more general. Second, a nonlinear term permits singularities with respect to both the time and the space variables.

2 Preliminaries and several lemmas

Let E be the Banach space of continuous functions $u : [0,1] \to R$ equipped with the norm $\|u\| = \max_{0 \le t \le 1} |u(t)|$. Here, we list some definitions and useful lemmas from fractional calculus theory.

Definition 1 ([3]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2 ([3]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to R$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Now, we consider the linear fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j). \end{cases} \tag{2}$$

Lemma 1 ([9]) *Given $y \in L^1[0,1]$, then the unique solution of the problem* (2) *can be expressed by*

$$u(t) = \int_0^1 G(t,s)y(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}, & 0 \le t \le s \le 1, \end{cases} \tag{3}$$

where $p(s) = \Delta - \sum_{s\le\xi_j} \alpha_j (\frac{\xi_j - s}{1-s})^{\alpha-1}(1-s)^i$. Obviously, $G(t,s)$ is continuous on $[0,1] \times [0,1]$.

Proof The proof is similar to that in [9] except for the convergence of the integral $\int_0^1 G(t,s)y(s)\,\mathrm{d}s$, which is easy to show. We omit it here. □

Lemma 2 ([7]) *The function $G(t,s)$ defined by* (3) *has the following properties*:
(1) $p(0)\Gamma(\alpha)G(t,s) \ge m_1 s(1-s)^{\alpha-1-i}t^{\alpha-1}$, $\forall t,s \in [0,1]$;
(2) $p(0)\Gamma(\alpha)G(t,s) \le [M_1 + p(0)n](1-s)^{\alpha-1-i}t^{\alpha-1}$, $\forall t,s \in [0,1]$;
(3) $p(0)\Gamma(\alpha)G(t,s) \le [M_1 + p(0)n]s(1-s)^{\alpha-1-i}$, $\forall t,s \in [0,1]$;
(4) $G(t,s) > 0$, $\forall t,s \in (0,1)$,
where $M_1 = \sup_{0<s\le1} \frac{p(s)-p(0)}{s}$, $m_1 = \inf_{0<s\le1} \frac{p(s)-p(0)}{s}$ are positive numbers.

Proof The proof of (1) and (3) is almost as the same as that in [7] and (4) is obvious. To get (2), check the proof of Lemma 2.4 in [7]. For $0 < s \le t \le 1$, we get

$$\begin{aligned} p(0)\Gamma(\alpha)G(t,s) &= p(s)(1-s)^{\alpha-1-i}t^{\alpha-1} - p(0)(t-s)^{\alpha-1} \\ &= \bigl[p(s) - p(0)\bigr](1-s)^{\alpha-1-i}t^{\alpha-1} + p(0)\bigl[(1-s)^{\alpha-1-i}t^{\alpha-1} - (t-s)^{\alpha-1}\bigr] \\ &\le M_1 s(1-s)^{\alpha-1-i}t^{\alpha-1} + p(0)(1-s)^{\alpha-1-i}t^{\alpha-1}\left[1 - \left(1 - \frac{s}{t}\right)\right]\left[1 + \left(1 - \frac{s}{t}\right)\right. \\ &\quad \left. + \left(1 - \frac{s}{t}\right)^2 + \cdots + \left(1 - \frac{s}{t}\right)^{n-1}\right] \\ &\le M_1 s(1-s)^{\alpha-1-i}t^{\alpha-1} + p(0)(1-s)^{\alpha-1-i}t^{\alpha-2}sn \\ &\le M_1 s(1-s)^{\alpha-1-i}t^{\alpha-1} + p(0)(1-s)^{\alpha-1-i}t^{\alpha-2}tn \\ &\le \bigl[M_1 + p(0)n\bigr](1-s)^{\alpha-1-i}t^{\alpha-1}. \end{aligned}$$

For $0 < t \le s \le 1$, we have

$$\begin{aligned} p(0)\Gamma(\alpha)G(t,s) &= p(s)(1-s)^{\alpha-1-i}t^{\alpha-1} \\ &= \bigl[p(s) - p(0)\bigr](1-s)^{\alpha-1-i}t^{\alpha-1} + p(0)(1-s)^{\alpha-1-i}t^{\alpha-1} \\ &\le \bigl[M_1 + p(0)n\bigr](1-s)^{\alpha-1-i}t^{\alpha-1}. \end{aligned}$$

Let $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then $\varphi_p^{-1}(s) = \varphi_q(s)$. To study the PFDE (1), we first consider the associated linear PFDE,

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + y(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\ D_{0+}^{\alpha}u(0) = 0,\ u^{(i)}(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j), \end{cases} \tag{4}$$

for $y \in L^1[0,1]$ and $y \geq 0$. □

Lemma 3 *The unique solution for the associated linear PFDE* (4) *can be written*

$$u(t) = \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} y(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Proof Let $w = D_{0+}^{\alpha}u$, $v = \varphi_p(w)$. Then the initial value problem

$$\begin{cases} D_{0+}^{\beta}v(t) + y(t) = 0, & t \in (0,1), \\ v(0) = 0 \end{cases} \tag{5}$$

has the solution $v(t) = c_1 t^{\beta-1} - I^{\beta}y(t)$, $t \in [0,1]$. Noticing that $v(0) = 0$, $0 < \beta \leq 1$, we have $c_1 = 0$. As a consequence,

$$v(t) = -I^{\beta}y(t), \quad t \in [0,1]. \tag{6}$$

Considering that $D_{0+}^{\alpha}u = w$, $w = \varphi_p^{-1}(v)$, we have from (6)

$$\begin{cases} D_{0+}^{\alpha}u(t) = \varphi_p^{-1}(-I^{\beta}(y(t))), & 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & u^{(i)}(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j). \end{cases} \tag{7}$$

By Lemma 1, the solution of (7) can be expressed by

$$u(t) = \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} y(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$ □

Definition 3 A continuous function $\Psi(t)$ is called a lower solution of the PFDE (1) if it satisfies

$$\begin{cases} -D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}\Psi(t))) \leq f(t,\Psi(t)), & 0 < t < 1, \\ \Psi(0) \geq 0,\ \Psi'(0) \geq 0,\ \ldots,\ \Psi^{(n-2)}(0) \geq 0,\ D_{0+}^{\alpha}\Psi(0) \geq 0,\ \Psi^{(i)}(1) \geq \sum_{j=1}^{\infty} \alpha_j \Psi(\xi_j). \end{cases}$$

Definition 4 A continuous function $\Phi(t)$ is called an upper solution of the PFDE (1) if it satisfies

$$\begin{cases} -D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}\Phi(t))) \geq f(t,\Phi(t)), & 0 < t < 1, \\ \Phi(0) \leq 0,\ \Phi'(0) \leq 0,\ \ldots,\ \Phi^{(n-2)}(0) \leq 0,\ D_{0+}^{\alpha}\Phi(0) \leq 0,\ \Phi^{(i)}(1) \leq \sum_{j=1}^{\infty} \alpha_j \Phi(\xi_j). \end{cases}$$

Let

$$F = \left\{ u \in C\bigl([0,1],R\bigr), u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\ u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j) \right\}.$$

Lemma 4 *Let $u \in F$ such that $-D_{0+}^{\alpha}u(t) \geq 0$, $t \in [0,1]$. Then $u(t) \geq 0$, $t \in [0,1]$.*

Proof Let $-D_{0+}^{\alpha}u(t) = h(t)$, here $h(t) \geq 0$, $t \in [0,1]$. Noticing that $u \in F$, by Lemma 1, we know that

$$u(t) = \int_0^1 G(t,s)h(s)\,\mathrm{d}s.$$

It follows from Lemma 2 and $h(t) \geq 0$ that $u(t) \geq 0$, $t \in [0,1]$. □

Lemma 5 (Leray-Schauder fixed point theorem) *Let T be a continuous and compact mapping of a Banach space E into itself, such that the set*

$$\{x \in E : x = \sigma Tx, \textit{for some } 0 \leq \sigma \leq 1\} \tag{8}$$

is bounded. Then T has a fixed point.

3 Main results

Denote $e(t) = t^{\alpha-1}$, $\overline{m} = \frac{m_1}{p(0)\Gamma(\alpha)}$, $\overline{M} = \frac{M_1+p(0)n}{p(0)\Gamma(\alpha)}$. We list below some assumptions used in this paper.

(H_0) $0 < \int_0^1 \varphi_p^{-1}(\int_0^s (s-\tau)^{\beta-1} f(\tau, e(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s < +\infty$.

(H_1) $f \in C((0,1) \times J, R^+)$, for any fixed $t \in (0,1)$, $f(t,u)$ is non-increasing in u, for any $c \in (0,1)$, there exists $\lambda > 0$ such that for all $(t,u) \in (0,1] \times J$,

$$f(t,cu) \leq c^{-\lambda} f(t,u). \tag{9}$$

From (9), it is easy to see that if $c \in [1,+\infty)$, then

$$f(t,cu) \geq c^{-\lambda} f(t,u). \tag{10}$$

Let

$$P = \{x \in C[0,1] : x(t) \geq 0, t \in [0,1]\}.$$

Obviously, P is a normal cone in the Banach space E. Now, define a subset D in E as follows:

$$\begin{aligned} D = \{&u \in P : \text{there exist two positive numbers } l_u < 1 < L_u \text{ such that} \\ &l_u e(t) \leq u(t) \leq L_u e(t), t \in [0,1]\}. \end{aligned} \tag{11}$$

Obviously, D is nonempty since $e(t) \in P$. Now define an operator A as follows:

$$(Au)(t) = \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0,1]. \tag{12}$$

Theorem 1 *Assume that* (H_0) *and* (H_1) *hold. Then the PFDE* (1) *has at least one positive solution $w^* \in D$, and there exist constants $0 < k < 1$ and $K > 1$ such that $ke(t) \leq w^*(t) \leq Ke(t)$, $t \in [0,1]$.*

Proof First, we show that $A : D \to D$ is well defined.

In fact, for any $u \in D$, there exist two positive numbers $L_u > 1 > l_u$ such that

$$l_u e(t) \le u(t) \le L_u e(t), \quad t \in [0,1]. \tag{13}$$

We have from (H_0), (H_1), Lemma 2, (9), (10), and (13)

$$\begin{aligned}
&\left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad \le \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \overline{M} t^{\alpha-1} \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, l_u e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad \le \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} l_u^{-\lambda(q-1)} \overline{M} \cdot \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \cdot e(t) \\
&\quad < \left[\left(\frac{1}{\Gamma(\beta)}\right)^{q-1} l_u^{-\lambda(q-1)} \overline{M} \cdot \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s + 1\right] e(t) \\
&\quad = L_u^* e(t)
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad \ge \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} L_u^{-\lambda(q-1)} \overline{m} \cdot \int_0^1 s(1-s)^{\alpha-1-i} \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \cdot e(t) \\
&\quad \ge l_u^* e(t),
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
l_u^* &= \min\left\{\frac{1}{2}, \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} L_u^{-\lambda(q-1)} \overline{m} \cdot \int_0^1 s(1-s)^{\alpha-1-i} \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\right\}, \\
L_u^* &= \left[\left(\frac{1}{\Gamma(\beta)}\right)^{q-1} l_u^{-\lambda(q-1)} \overline{M} \cdot \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s + 1\right].
\end{aligned}$$

By (H_0), it is clear that

$$\begin{aligned}
&\int_0^1 s(1-s)^{\alpha-1-i} \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad \le \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s < +\infty.
\end{aligned}$$

Thus, from (14) and (15), we know that $A : D \to D$ and is well defined.

By Lemma 3, we know that $Au(t)$ satisfies the following equation:

$$\begin{cases}
-D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}(Au)(t))) = f(t,u(t)), \quad 0 < t < 1, \\
(Au)(0) = (Au)'(0) = \cdots = (Au)^{(n-2)}(0) = 0, \\
D_{0+}^{\alpha}(Au)(0) = 0, \qquad (Au)^{(i)}(1) = \sum_{j=1}^{\infty} \eta_j (Au)(\xi_j).
\end{cases} \tag{16}$$

Now, we are in a position to find a pair of upper and lower solutions for PFDE (1). Let

$$u_0(t) = \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0,1].$$

By Lemma 2, we get

$$u_0(t) \geq \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \overline{m} \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \cdot e(t), \quad t \in [0,1].$$

As a consequence, there exists a constant $k_0 \geq 1$ such that

$$k_0 u_0(t) \geq e(t), \quad \forall t \in [0,1]. \tag{17}$$

It follows from (H_0), (H_1), and (17) that A is decreasing on u, thus for $k > k_0$, we have

$$\begin{aligned}
&\left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, k u_0(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad \leq \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, k_0 u_0(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad \leq \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s < +\infty
\end{aligned}$$

and

$$u_0(t) \leq \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \overline{M} \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s < +\infty.$$

Let $\rho = (\frac{1}{\Gamma(\beta)})^{q-1}\overline{M}\int_0^1 \varphi_p^{-1}(\int_0^s (s-\tau)^{\beta-1} f(\tau, e(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s + 1$. Take

$$k^* = \max\left\{k_0, \left[\left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \overline{m} \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, \rho)\,\mathrm{d}\tau\right)\mathrm{d}s\right]^{\frac{1}{\lambda(q-1)}}\right\}.$$

Then we have

$$\begin{aligned}
+\infty &> \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, k^* u_0(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq \big(k^*\big)^{-\lambda(q-1)}\left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \overline{m} t^{\alpha-1} \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f\big(\tau, u_0(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq \big(k^*\big)^{-\lambda(q-1)}\left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \overline{m} t^{\alpha-1} \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1} f(\tau, \rho)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq t^{\alpha-1}, \quad \forall t \in [0,1].
\end{aligned} \tag{18}$$

Let

$$\Phi(t) = k^* u_0(t), \qquad \Psi(t) = (A\Phi)(t). \tag{19}$$

Then it follows from (17) and (18) that

$$\Phi(t) = k^*\left(\frac{1}{\Gamma(\beta)}\right)^{q-1}\int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1}f\big(\tau,e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \geq t^{\alpha-1}, \tag{20}$$

$$\Psi(t) = \left(\frac{1}{\Gamma(\beta)}\right)^{q-1}\int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1}f\big(\tau,k^*u_0(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \geq t^{\alpha-1}. \tag{21}$$

In addition, by (16) and (19), we see that

$$\Phi(0) = \Phi'(0) = \cdots = \Phi^{(n-2)}(0) = 0, \qquad D_{0+}^{\alpha}\Phi(0) = 0, \qquad \Phi^{(i)}(1) = \sum_{j=1}^{\infty}\eta_j\Phi(\xi_j),$$

$$\Psi(0) = \Phi'(0) = \cdots = \Psi^{(n-2)}(0) = 0, \qquad D_{0+}^{\alpha}\Psi(0) = 0, \qquad \Psi^{(i)}(1) = \sum_{j=1}^{\infty}\eta_j\Psi(\xi_j).$$

By (19),

$$\begin{aligned}\Psi(t) &= (A\Phi)(t) = \left(\frac{1}{\Gamma(\beta)}\right)^{q-1}\int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1}f\big(\tau,k^*u_0(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\\ &\leq k^*\left(\frac{1}{\Gamma(\beta)}\right)^{q-1}\int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1}f\big(\tau,u_0(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\\ &= \Phi(t), \quad \forall t\in[0,1].\end{aligned} \tag{22}$$

Considering the fact that f is non-increasing in u, we see from (19)-(21) that

$$\begin{aligned}D_{0+}^{\beta}\big(\varphi_p\big(D_{0+}^{\alpha}\Psi(t)\big)\big)+f\big(t,\Psi(t)\big) &= D_{0+}^{\beta}\big(\varphi_p\big(D_{0+}^{\alpha}(A\Phi)(t)\big)\big)+f\big(t,\Psi(t)\big)\\ &\geq D_{0+}^{\beta}\big(\varphi_p\big(D_{0+}^{\alpha}(A\Phi)(t)\big)\big)+f\big(t,\Phi(t)\big)\\ &= -f\big(t,\Phi(t)\big)+f\big(t,\Phi(t)\big) = 0,\end{aligned} \tag{23}$$

$$\begin{aligned}D_{0+}^{\beta}\big(\varphi_p\big(D_{0+}^{\alpha}\Phi(t)\big)\big)+f\big(t,\Phi(t)\big) &= D_{0+}^{\beta}\big(\varphi_p\big(D_{0+}^{\alpha}A\big(t^{\alpha-1}\big)\big)\big)+f\big(t,\Phi(t)\big)\\ &= -f\big(t,t^{\alpha-1}\big)+f\big(t,\Phi(t)\big)\\ &\leq -f\big(t,t^{\alpha-1}\big)+f\big(t,t^{\alpha-1}\big) = 0.\end{aligned} \tag{24}$$

By (23) and (24), we know that $\Phi,\Psi\in P$ are the desired upper and lower solutions of the PFDE (1), respectively.

Define a function F as follows:

$$F(t,u) = \begin{cases} f(t,\Psi(t)), & u<\Psi(t),\\ f(t,u(t)), & \Psi(t)\leq u\leq\Phi(t),\\ f(t,\Phi(t)), & \Phi(t)<u.\end{cases} \tag{25}$$

This together with (H_1) shows that $F:(0,1)\times R^+\to R^+$ is continuous.

In the following, we shall show that the fractional boundary value problem

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t)))+F(t,u(t)) = 0, \quad 0<t<1,\\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\ D_{0+}^{\alpha}u(0) = 0,\ u^{(i)}(1) = \sum_{j=1}^{\infty}\alpha_j u(\xi_j),\end{cases} \tag{26}$$

has a positive solution.

Let

$$(Tu)(t) = \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1}F\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s, \quad t \in [0,1]. \tag{27}$$

Then $T : E \to E$ and a fixed point of the operator T is a solution of the PFDE (26). By (20), (21), the definition of F, and the fact that $f(t,u)$ is non-increasing in u, we have

$$f\big(t,\Phi(t)\big) \leq F\big(t,u(t)\big) \leq f\big(t,\Psi(t)\big), \quad \forall x \in E, \tag{28}$$

and

$$f\big(t,\Phi(t)\big) \leq F\big(t,u(t)\big) \leq f\big(t,t^{\alpha-1}\big), \quad \forall x \in E. \tag{29}$$

By Lemma 2 and (29), for $u \in E$, we have

$$\begin{aligned}(Tu)(t) &= \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \int_0^1 G(t,s)\varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1}F\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\leq \left(\frac{1}{\Gamma(\beta)}\right)^{q-1} \overline{M}t^{\alpha-1} \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1}f\big(\tau,e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s < +\infty,\end{aligned} \tag{30}$$

which means that T is uniformly bounded. Considering the uniform continuity of $G(t,s)$ on $[0,1] \times [0,1]$, it can easily be seen that $T : E \to E$ is completely continuous. In addition, we see from (30) that (8) holds. Thus, Schauder fixed point theorem guarantees that T has at least one fixed point w.

Now, we are in a position to show that

$$\Psi(t) \leq w(t) \leq \Phi(t), \quad t \in [0,1]. \tag{31}$$

Since w is a fixed point of T, we have by (26)

$$w(0) = w'(0) = \cdots = w^{(n-2)}(0) = 0, \qquad D_{0+}^{\alpha}w(0) = 0, \qquad w^{(i)}(1) = \sum_{j=1}^{\infty}\alpha_j w(\xi_j), \tag{32}$$

$$\Phi(0) = \Phi'(0) = \cdots = \Phi^{(n-2)}(0) = 0, \qquad D_{0+}^{\alpha}\Phi(0) = 0, \qquad \Phi^{(i)}(1) = \sum_{j=1}^{\infty}\eta_j \Phi(\xi_j). \tag{33}$$

Let $z(t) = \varphi_p(D_{0+}^{\alpha}\Phi(t)) - \varphi_p(D_{0+}^{\alpha}w(t))$. Then

$$\begin{aligned}D_{0+}^{\beta}z(t) &= D_{0+}^{\beta}\big(\varphi_p\big(D_{0+}^{\alpha}\Phi(t)\big)\big) - D_{0+}^{\beta}\big(\varphi_p\big(D_{0+}^{\alpha}w(t)\big)\big) \\ &= -f\big(t,t^{\alpha-1}\big) + F\big(t,w(t)\big) \leq 0, \quad t \in [0,1], \\ z(0) &= \varphi_p\big(D_{0+}^{\alpha}\Phi(0)\big) - \varphi_p\big(D_{0+}^{\alpha}w(0)\big) = 0.\end{aligned}$$

By (5) and (6), we know that

$$z(t) \leq 0,$$

which means that

$$\varphi_p\big(D_{0+}^{\alpha}\Phi(t)\big) - \varphi_p\big(D_{0+}^{\alpha}w(t)\big) \le 0.$$

We see from the fact that φ_p is monotone increasing

$$\varphi_p\big(D_{0+}^{\alpha}\Phi(t)\big) \le \varphi_p\big(D_{0+}^{\alpha}w(t)\big), \quad i.e., -\varphi_p\big(D_{0+}^{\alpha}(\Phi - w)\big)(t) \ge 0.$$

It follows from Lemma 4, (32), and (33) that

$$\Phi(t) - w(t) \ge 0.$$

Thus, we have proved that $w(t) \le \Phi(t)$ on $[0,1]$. Similarly, we can get $w(t) \ge \Psi(t)$ on $[0,1]$. As a consequence, (31) holds. So, $F(t,w(t)) = f(t,w(t))$, $t \in [0,1]$. Hence, $w(t)$ is a positive solution of the PFDE (1). Noticing that $\Phi, \Psi \in D$, by (31), we can easily see that there exist constants $0 < k < 1$ and $K > 1$ such that

$$ke(t) \le w^*(t) \le Ke(t), \quad t \in [0,1]. \qquad \square$$

4 An example

Consider the following singular PFDE:

$$\begin{cases} D_{0+}^{\frac{1}{2}}(\varphi_3(D_{0+}^{\frac{7}{2}}u))(t) + \frac{1}{2}t^{-\frac{1}{12}}u^{-\frac{1}{6}} = 0, & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ D_{0+}^{\frac{7}{2}}u(0) = 0, \qquad u'(1) = \sum_{j=1}^{\infty}\frac{2}{j^2}u(\frac{1}{j}). \end{cases} \tag{34}$$

In this situation, $f(t,u) = \frac{1}{2}t^{-\frac{1}{12}}u^{-\frac{1}{6}}$, $\alpha = \frac{7}{2}$, $\beta = \frac{1}{2}$, $p = 3$, $e(t) = t^{\frac{5}{2}}$, $\Delta = \frac{5}{2}$, $\alpha_j = \frac{2}{j^2}$, $\xi_j = \frac{1}{j}$, $\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-1} \approx 2.109 < \Delta$. By a simple computation, we have

$$\begin{aligned} 0 &< \int_0^1 \varphi_p^{-1}\left(\int_0^s (s-\tau)^{\beta-1}f\big(\tau, e(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &= \int_0^1 \left(\frac{1}{2}\int_0^s (s-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{12}}\tau^{-\frac{5}{12}}\,\mathrm{d}\tau\right)^{\frac{1}{2}}\mathrm{d}s \\ &= \int_0^1 \left(\frac{1}{2}\int_0^1 (1-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\,\mathrm{d}\tau\right)^{\frac{1}{2}}\mathrm{d}s = \frac{\sqrt{2\pi}}{2} < +\infty. \end{aligned}$$

Therefore, (H_1) holds. It is easy to see that (H_2) is satisfied for $\lambda = \frac{1}{6}$. By Theorem 1, PFDE (34) has at least one positive solution w^* such that there exist constants $0 < k < 1$ and $K > 1$ with $ke(t) \le w^*(t) \le Ke(t)$, $t \in [0,1]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors, QZ and XZ, contributed to each part of this work equally and read and approved the final version of the manuscript.

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